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BOUNDARY SHAPE DISTORTION INVOLVING DIRICHLET CONDITIONS WITH APPLICATION TO TEMPERATURE DISTRIBUTION

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SUMMARY

This paper presents a variant of the method of separation of variables which enables the determination of the solution of an elliptic differential equation having Dirichlet conditions along an arbitrary curve D forming part of the boundary. The coefficients in the eigenfunction expansions representing the general solution are determined by comparison with a special series representation of the Dirichlet condition along D. This representation is obtained by means of the Gram-Schmidt orthogonalization process which uses as its basic non-orthogonal set of functions a special set derived directly from the eigenfunction expansion. A simple numerical example concerning the temperature distribution in a semi-infinite parallel slab with a skew end face on which there is a sinusoidal temperature varation illustrates the application of the method. It is shown that the rate of convergence is good and that the asymptotic solution is estimated rapidly and accurately by this method.

1. *Introduction.*

It sometimes happens that an analytical rather than a numerical solution is required to a linear elliptic differential equation that is subject to some general boundary conditions over part of the boundary which is described by attributing constant values to coordinate variables, and to Dirichlet conditions along the remainder of the boundary which does not coincide with a constant value of a coordinate variable. Under these circumstances the method of separation of variables is not directly applicable and recourse must be made to an approximate method of solution such as the boundary shape perturbation method described by Morse and Feshbach [1]. Unfortunately the rate of convergence of this method is usually rather poor when non-trivial boundary shape perturbations are involved so that the accompanying manipulative work becomes extremely tedious.

The present paper describes a simple variant of the method of separation of variables which can often be used to advantage in these circumstances. The accuracy with which the solution is represented is shown to depend on the degree to which the boundary condition on the distorted section of the boundary is approximated by a partial sum formed from a certain complete orthogonal sequence of functions. Here the word *distorted* is used to describe that part of the boundary which does not coincide with a constant value of a coordinate variable.

Determination of the coefficients in the expansion representing the solution involves simple quadratures which can often be performed analytically, though when with complicated shapes numerical quadrature becomes necessary the ensuing task is always much less onerous than the corresponding calculations for the perturbation solution since these necessitate multiple quadratures.

By way of illustration a simple application of the method is made to the problem of the steady state temperature distribution in a semi-infinite slab of material of finite width, the parallel sides of which are maintained at zero temperature whilst a skew end surface is subjected to a sinusoidal temperature distribution with amplitude T_1. Finally the maximum principle for elliptic differential equations is used to place bounds on the asymptotic behaviour of the solution which may be deduced directly from the approximate solution. From the results it may be concluded that the asymptotic component of the approximate solution converges rapidly to the true result.

2. *General method*

To explain the general method it will be sufficient for us to consider a function $T(x,y)$, of the independent variables x and y, which satisfies Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.1}$$

in some region R, subject to some general boundary conditions on part D_1 of the boundary and to Dirichlet conditions on the remainder D_2 of the boundary, where D_1 and D_2 are defined as follows.

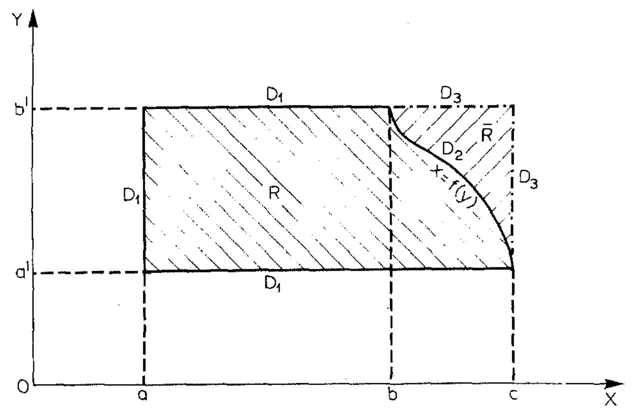

Fig. 1. Distorbed Boundary Shape

The boundary D_1 comprises the three line segments:

$$\left.\begin{array}{l} a \leq x < b \;\; ; \; y = b' \\ a' < x < b' \;\; ; \; x = a \\ a \leq x < c \;\; ; \; y = a' \end{array}\right\} \tag{2.2}$$

whilst the boundary D_2 is defined by the rectifiable arc

$$x = f(y) \; ; \; a' \leq y \leq b', \tag{2.4}$$

along which

$$T(f(y), y) = T_1(y) \tag{2.4}$$

is some given function.

Let us now supplement the region R by the addition of the region $\bar{R}$ (See Fig. 1.) contained between the arc D_2 and line segments D_3 defined by:

$$\left.\begin{array}{l} b \leq x < c \;\; ; \; y = b' \\ a' \leq y \leq b' \;\; ; \; x = c \end{array}\right\} \tag{2.5}$$

We now determine the solution in the region $R + \bar{R}$ by continuing the boundary condition on $a \leq x < b$; $y = b'$ to the segment $b \leq x < c$; $y = b'$ in such a manner that T assumes the specified boundary values along D_2. This is, of course, an unusually posed boundary value problem in the sense that until the problem is solved, the boundary conditions along the remainder of D_3 are *unknown*.

Separating the variables in equation (2.1) by writing

$$T(x,y) = X(x)\ Y(y), \tag{2.6}$$

and introducing the separation constant α in the usual manner through the relation

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha^2, \tag{2.7}$$

then leads to the result

$$Y = A \cos \alpha y + B \sin \alpha y. \tag{2.8}$$

Then, by use of the boundary conditions to be imposed along $a \leq x < b$; $y = b'$ and $a \leq x < c$; $y = a'$, a set of eigenvalues $\alpha_1, \alpha_2, \ldots$ may be deduced from equation (2.8) giving rise to the eigenfunctions $Y_n(y)$ of the form

$$Y_n(y) = k_{1n} \cos \alpha_n y + k_{2n} \sin \alpha_n y, \tag{2.9}$$

where k_{1n} and k_{2n} are known constants. The corresponding eigenfunctions $X_n(x)$ are

$$X_n(x) = a_n e^{\alpha_n x} + b_n e^{-\alpha_n x}, \tag{2.10}$$

showing that the general solution $T(x,y)$ will be of the form

$$T(x,y) = \sum_{n=1}^{\infty} (a_n e^{\alpha_n x} + b_n e^{-\alpha_n x})\ (k_{1n} \cos \alpha_n y + k_{2n} \sin \alpha_n y), \tag{2.11}$$

Now, by applying the usual Fourier method [1, Chapter 6] to the boundary condition along $a' < y < b'$; $x = a$ one relationship may readily obtained between the a_n and b_n so that (2.11) then takes the form

$$T(x,y) = \sum_{n=1}^{\infty} c_n(\ell_{1n} e^{\alpha_n x} + \ell_{2n} e^{-\alpha_n x})(k_{1n} \cos \alpha_n y + k_{2n} \sin \alpha_n y), \tag{2.12}$$

where the constants ℓ_{1n} and ℓ_{2n} are known, and it remains to determine the constants c_n. This will be achieved by utilizing the fact that when $x = f(y)$; $a' \leq y \leq b'$, it follows from the boundary condition (2.4) that

$$T_1(y) = \sum_{n=1}^{\infty} c_n \{\ell_{1n} \exp(\alpha_n f(y)) + \ell_{2n} \exp(-\alpha_n f(y))\} \times$$
$$\times (k_{1n} \cos \alpha_n y + k_{2n} \sin \alpha_n y). \tag{2.13}$$

However, before proceeding with the determination of these constants, let us first remark that if the boundary condition were known on D_3 such that expression (2.11) assumed the required values along the arc D_2 then, for all physically real boundary conditions, the series (2.11), and hence the series (2.13) would be uniformly convergent in R and $\bar{R}$. Consequently, re-arrangement of the terms of expression (2.12) is permissible.

Accordingly then, let us utilize the completeness of the sequences of eigenfunctions $X_n(x)$ and $Y_n(y)$, and the fact that re-arrangement of terms is permissible, to begin by forming a complete orthogonal sequence $\Phi_n(y)$ from the functions $z_n(y)$ by means of the Gram-Schmidt method [1, 2], where

$$z_n(y) = \{\ell_{1n} \exp(\alpha_n f(y)) + \ell_{2n} \exp(-\alpha_n f(y))\} (k_{1n} \cos \alpha_n y + k_{2n} \sin \alpha_n y). \tag{2.14}$$

Defining the inner product of two functions $h(y)$ and $k(y)$ on the interval $a' \leq y \leq b'$ by

$$(h, k) = \int_{a'}^{b'} h(y)\, k(y)\, dy$$

we immediately arrive at the *complete orthogonal sequence* $\Phi_n(y)$,

$$\Phi_n(y) = z_n(y) - \sum_{k=1}^{n-1} \frac{(z_n, \Phi_k)}{(\Phi_k, \Phi_k)} \Phi_k(y). \tag{2.15}$$

Hence, re-writing series (2.13) in the form

$$T_1(y) = \sum_{n=1}^{\infty} d_n \Phi_n(y), \tag{2.16}$$

it follows by the orthogonality of the sequence of functions $\Phi_n(y)$ that

$$d_n = \frac{(T_1, \Phi_n)}{(\Phi_n, \Phi_n)}. \tag{2.17}$$

Now since, by virtue of equations (2.15), the functions $\Phi_n(y)$ are known in terms of the functions $z_n(y)$, the partial sum approximation $s_N(y)$ to $T_1(y)$ which is given by

$$s_N(y) = \sum_{n=1}^{N} d_n \Phi_n(y), \tag{2.18}$$

may be re-written in the form

$$s_N(y) = \sum_{n=1}^{N} c_n^{(N)} z_n(y) \tag{2.19}$$

Identification of the coefficients $c_n^{(N)}$ with the coefficients c_n of the corresponding functions appearing in expression (2.13) then determines the approximations $c_n^{(N)}$ to the first N constants c_n. Utilizing these values in series (2.12) will give the partial sum approximation to the general solution $T(x, y)$ which is obtained by using the first N functions $\Phi_n(y)$.

Clearly, as the distorted boundary D_2 tends to the associated undistorted boundary D_3, so the solution obtained by this method tends to the known solution applicable to the undistorted boundary with the boundary conditions along D_3 set equal to those along D_2.

3. *Temperature distribution in a semi-infinite plane slab with a skew end surface*

By way of illustration we now apply the method of the previous section to a simple problem involving steady state temperature distribution. We consider a semi-infinite plane slab of material of unit thickness whose side faces are maintained at zero temperature whilst a skew end surface is subjected to a sinusoidal temperature distribution with amplitude T_1. A typical cross-section of the slab is illustrated in Fig.2 in which the

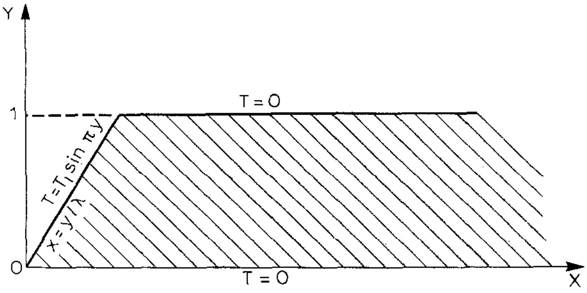

Fig. 2. Semi-Infinite Slab With Skew End Surface

skew end surface is defined by the equation $x = y/\lambda$, $0 \leq y \leq 1$.

Separating the variables in equation (2.1) as indicated, and using the boundary condition $T = 0$ on boundary surfaces $y = 0$ and $y = 1$, $x > 0$, together with the fact that the solution $T(x,y)$ must always remain finite, leads directly to the general solution

$$T(x,y) = \sum_{n=1}^{\infty} c_n e^{-n\pi x} \sin n\pi y \tag{3.1}$$

which corresponds to equation (2.12).

Since $T = T_1 \sin \pi y$ along $x = y/\lambda$, $0 \leq y \leq 1$, we then have that

$$T_1 \sin \pi y = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi y}{\lambda}} \sin n\pi y, \tag{3.2}$$

so that in this case the functions $z_n(y)$ corresponding to (2.14) are seen to be

$$z_n(y) = e^{\frac{-n\pi y}{\lambda}} \sin n\pi y \ , \ n = 1, 2, \ldots$$

It is easily verified that the inner products

$$(z_n, z_n) = \int_0^1 z_n^2 (y)\, dy \text{ and } (z_m, z_n) = \int_0^1 z_m (y)\, z_n (y)\, dy$$

respectively have the values:

$$(z_n, z_n) = \frac{\lambda(1 - e^{\frac{-2n\pi}{\lambda}})}{4n\pi(1 + 1/\lambda^2)} \tag{3.3}$$

and

$$(z_m, z_n) = \frac{\left\{1 + (-1)^{m+n+1} . e^{\frac{-(m+n)\pi}{\lambda}}\right\}}{2\pi\lambda} \left[\frac{(m+n)}{(m-n)^2 + (m+n)^2/\lambda^2} + - \frac{1}{(m+n)(1+1/\lambda^2)}\right]. \tag{3.4}$$

Using these results together with equation (2.15) then gives rise to an orthogonal sequence of functions $\Phi_n(y)$ on the interval $0 \leq y \leq 1$. The coefficients d_n corresponding to (2.17), and hence the approximations $c_n^{(N)}$ to c_n in the N-th approximation then follow directly the $\phi_n(y)$ have been determined. The final steps in the calculation can best be illustrated by the computation of specific partial sum approximations which will be accomplished in the next section.

4. *Numerical details of* $s_3(y)$, $s_4(y)$ *and* $s_5(y)$ *approximations*

If $T(x, y)$ is to be successively approximated by utilizing the $s_3(y)$, $s_4(y)$ and $s_5(y)$ approximations to the boundary condition along the distorted boundary corresponding to the skew face, the m and n of the previous section must be allowed to assume the values 1 to 5. Simple calculations establish the following results for a skew end surface defined by the equation $x = 0.1y$ (i.e. $\lambda = 10$):

Table of (z_i, z_j)

j \ i	1	2	3	4	5
1	0.3674	0.0536	0.0081	0.0066	0.0028
2	0.0536	0.2818	0.0731	0.0164	0.0105
3	0.0081	0.0731	0.2227	0.0806	0.0234
4	0.0066	0.0164	0.0806	0.1810	0.0820
5	0.0028	0.0105	0.0234	0.0820	0.1508

Then, from (2.15) the first five orthogonal functions $\Phi_n(y)$ are

$$\left.\begin{aligned}
\Phi_1(y) &= z_1(y),\\
\Phi_2(y) &= z_2(y) - 0.1458\ z_1(y),\\
\Phi_3(y) &= z_3(y) - 0.2623\ z_2(y) + 0.0161\ z_1(y),\\
\Phi_4(y) &= z_4(y) - 0.3749\ z_3(y) + 0.0420\ z_2(y)\\
&\qquad - 0.0158\ z_1(y),\\
\Phi_5(y) &= z_5(y) - 0.4858\ z_4(y) + 0.0808\ z_3(y)\\
&\qquad - 0.0306\ z_2(y) + 0.0039\ z_1(y),
\end{aligned}\right\} \tag{4.1}$$

from which it follows that

$$(\Phi_1, \Phi_1) = 0.3674, \quad (\Phi_2, \Phi_2) = 0.2741,$$
$$(\Phi_3, \Phi_3) = 0.2038, \quad (\Phi_4, \Phi_4) = 0.1514,$$
$$(\Phi_5, \Phi_5) = 0.1126.$$

Since $T = T_1 \sin\pi y$ along the skew face the numerator of (2.17) simplifies to

$$(T, \Phi_n) = T_1 \int_0^1 \sin\pi y\ \Phi_n(y)\ dy.$$

Using the form of $z_n(y)$ together with equations (4.1) a simple calculation shows that

$$(T, \Phi_1) = 0.4290T_1, \quad (T, \Phi_2) = -0.0199T_1,$$
$$(T, \Phi_3) = 0.0013T_1, \quad (T, \Phi_4) = -0.0013T_1,$$
$$(T, \Phi_5) = -0.0012T_1.$$

The desired approximations $s_3(y)$, $s_4(y)$ and $s_5(y)$ to $T_1 \sin\pi y$ along the skew face $x = 0.1y$, $0 \le y \le 1$ are thus

$$s_3(y) = T_1[1.1650\ \Phi_1(y) - 0.0728\ \Phi_2(y) + 0.0065\ \Phi_3(y)], \tag{4.2}$$

$$s_4(y) = T_1[1.1560\ \Phi_1(y) - 0.0728\ \Phi_2(y) + 0.0065\ \Phi_3(y) - 0.0088\ \Phi_4(y)], \tag{4.3}$$

$$s_5(y) = T_1 [1.1560\ \Phi_1(y) - 0.0728\ \Phi_2(y) + 0.0065\ \Phi_3(y) - 0.0088\ \Phi_4(y) - 0.0116\ \Phi_5(y)].$$

Using equations (4.1) gives the alternative expressions in terms of $z_n(y)$:

$$s_3(y) = T_1 [1.1773 z_1(y) - 0.0845 z_2(y) + 0.0065 z_3(y)], \tag{4.5}$$

$$s_4(y) = T_1 [1.1774 z_1(y) - 0.0749 z_2(y) + 0.0098 z_3(y) - 0.0088 z_4(y)], \tag{4.6}$$

$$s_5(y) = T_1 [1.1774 z_1(y) - 0.0745 z_2(y) + 0.0089 z_3(y) - 0.0032 z_4(y) - 0.0116 z_5(y)]. \tag{4.7}$$

Identification of the coefficients of (4.5) to (4.7) with the corresponding ones of (3.2) gives the results:

Case N = 3

$$c_1^{(3)} = 1.1774 T_1,\quad c_2^{(3)} = -0.0745 T_1,\quad c_3^{(3)} = 0.0065 T_1$$

Case N = 4

$$c_1^{(4)} = 1.1774 T_1,\quad c_2^{(4)} = -0.0746 T_1,\quad c_3^{(4)} = 0.0098 T_1,\quad c_4^{(4)} = -0.0088 T_1$$

Case N = 5

$$c_1^{(5)} = 1.1774 T_1,\quad c_2^{(5)} = -0.0749 T_1,\quad c_3^{(5)} = 0.0089 T_1,\quad c_4^{(5)} = -0.0032 T_1,\quad c_5^{(5)} = -0.0116 T_1.$$

The T_3, T_4 and T_5 approximations to $T(x,y)$ are thus represented by

$$T_3(x,y) = T_1[1.1773 e^{-\pi x} \sin\pi y - 0.0745 e^{-2\pi x} \sin 2\pi y + 0.0065 e^{-3\pi x} \sin 3\pi y], \tag{4.8}$$

$$T_4(x,y) = T_1[1.1774 e^{-\pi x} \sin\pi y - 0.0746 e^{-2\pi x} \sin 2\pi y + 0.0098 e^{-3\pi x} \sin 3\pi y - 0.0088 e^{-4\pi x} \sin 4\pi y] \tag{4.9}$$

$$T_5(x,y) = T_1[1.1774 e^{-\pi x} \sin\pi y - 0.0749 e^{-2\pi x} \sin 2\pi y + 0.0089 e^{-3\pi x} \sin 3\pi y - 0.0032 e^{-4\pi x} \sin 4\pi y - 0.0116 e^{-5\pi x} \sin 5\pi y]. \tag{4.10}$$

5. *Comparison of result with undistorted case*

It is interesting to compare the results of the previous section with the comparison solution $T_c(x,y)$ obtained directly by the method of separation of variables when the boundary is undistorted, so that the end surface is normal to the sides of the slab (i.e. the case $\lambda \longrightarrow \infty$). This comparison solution may then be used together with the maximum principle for elliptic equations [2] to place bounds on the approximations $T_3(x,y)$ to $T_5(x,y)$ to the genuine solution $T(x,y)$. In particular, these may also be used to estimated the accuracy of the approximations to the true asymptotic behaviour.

The solution in the undistorted case which is readily seen to be

$$T_c(x,y) = T_1\, e^{-\pi x} \sin y, \tag{5.1}$$

requires some modification before a useful comparison may be made with $T_3(x, y)$ to $T_5(x, y)$. By the maximum principle for elliptic differential equations, if a sinusoid distribution of amplitude T_1 is assumed at $x = 0$ then the resulting solution will *underestimate* $T(x, y)$ at corresponding points (x, y); whereas if it is assumed on the plane $x = 0.1$, then the resulting solution will *overestimate* $T(x, y)$ at corresponding points for $x \geq 0.1$.

We thus deduce that

$$T_1 e^{-\pi x} \sin\pi y < T(x, y) < T_1 e^{-\pi(x-0.1)} \sin\pi y$$

or equivalently, for $x > 0.1$,

$$T_1 e^{-\pi x} \sin\pi y < T(x, y) < 1.3689 T_1 e^{-\pi x} \sin\pi y \tag{5.2}$$

The asymptotic term in the T_4 and T_5 approximations are identical and are seen to be given by

$$T_5^{(asy)}(x, y) = 1.1774 T_1 e^{-\pi x} \sin\pi y. \tag{5.3}$$

As this result is contained within the inequality (5.2) and, further, the coefficients $c_1^{(3)}$, $c_1^{(4)}$ and $c_1^{(5)}$ converged so rapidly, we may assume that the method rapidly approximates the asymptotic solution and that higher approximations serve to improve accuracy for small x.

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